

# From trivial spectrum subspaces to spaces of diagonalizable real matrices

Clément de Seguins Pazzis<sup>\*†</sup>

June 2, 2016

*Dedicated to Bernard Randé, on the occasion of his retirement.*

## Abstract

A recent generalization of Gerstenhaber's theorem on spaces of nilpotent matrices [5] is shown to yield a new proof of the classification of linear subspaces of diagonalizable real matrices with the maximal dimension [3].

*AMS Classification:* 15A30; 15A03

*Keywords:* trivial spectrum spaces, dimension, diagonalizable matrices, symmetric matrices, vector spaces of matrices.

## 1 Introduction

Let  $\mathbb{F}$  be an arbitrary field and  $n$  be a positive integer. We denote respectively by  $M_n(\mathbb{F})$ ,  $S_n(\mathbb{F})$  and  $A_n(\mathbb{F})$  the spaces of all  $n$  by  $n$  square matrices, symmetric matrices, and alternating matrices with entries in  $\mathbb{F}$ . We denote by  $GL_n(\mathbb{F})$  the group of all invertible matrices of  $M_n(\mathbb{F})$ .

A **trivial spectrum subspace** of  $M_n(\mathbb{F})$  is a linear subspace in which no matrix has a non-zero eigenvalue in the field  $\mathbb{F}$ . Such subspaces have attracted much focus in the recent past, in part due to their connection with affine subspaces of non-singular matrices, and also because they are a generalization of

---

<sup>\*</sup>Université de Versailles Saint-Quentin-en-Yvelines, Laboratoire de Mathématiques de Versailles, 45 avenue des États-Unis, 78035 Versailles cedex, France

<sup>†</sup>e-mail address: dsp.prof@gmail.com

Gerstenhaber's linear subspaces of nilpotent matrices [1]. A subset  $V$  of  $M_n(\mathbb{F})$  is called **irreducible** when no non-trivial linear subspace of  $\mathbb{F}^n$  is stable under all the elements of  $V$ . The main theorem on trivial spectrum subspaces then reads as follows (see [5] for the original proof, and [4] for a simplified proof over fields with large cardinality).

**Theorem 1** (de Seguins Pazzis (2013)). *Let  $V$  be an irreducible trivial spectrum subspace of  $M_n(\mathbb{F})$ . Then:*

- (a)  $\dim V \leq \binom{n}{2}$ .
- (b) If  $\dim V = \binom{n}{2}$  and  $|\mathbb{F}| > 2$  then  $V = P A_n(\mathbb{F})$  for some matrix  $P$  of  $GL_n(\mathbb{F})$  which is non-isotropic, i.e.  $\forall X \in \mathbb{F}^n \setminus \{0\}, X^T P X \neq 0$ .

In this short note, we shall uncover a surprising connection between trivial spectrum subspaces and large linear subspaces of diagonalizable matrices. The main theorem on the latter [3], which is especially relevant to the case of real matrices, reads as follows:

**Theorem 2** (Randé, de Seguins Pazzis (2011)). *Let  $V$  be a diagonalizable subspace of  $M_n(\mathbb{F})$ , i.e. a linear subspace in which every matrix is diagonalizable. Then :*

- (a)  $\dim V \leq \binom{n+1}{2}$ .
- (b) If  $\dim V = \binom{n+1}{2}$  then  $V$  is similar to  $S_n(\mathbb{F})$ , i.e.  $V = P S_n(\mathbb{F}) P^{-1}$  for some  $P \in GL_n(\mathbb{F})$ .

Statement (a) is readily obtained by noting that  $V$  is linearly disjoint from the linear subspace of all strictly upper-triangular matrices of  $M_n(\mathbb{F})$ .

Over an arbitrary field, not all symmetric matrices are diagonalizable (see [6]), and hence the maximal dimension for a diagonalizable subspace of  $M_n(\mathbb{F})$  might be less than  $\binom{n+1}{2}$ . In the extreme case when  $\mathbb{F}$  is algebraically closed with characteristic 0, the celebrated Motzkin-Taussky theorem [2] shows that the maximal dimension is  $n$ .

We shall prove that, provided that  $\mathbb{F}$  has more than 2 elements, point (b) in Theorem 2 is a rather simple consequence of Theorem 1.

## 2 Proof of Theorem 2

For  $(i, j) \in \llbracket 1, n \rrbracket^2$ , we denote by  $E_{i,j}$  the matrix unit of  $M_n(\mathbb{F})$  in which every entry equals zero except the one at the  $(i, j)$ -spot, which equals 1.

The key idea is to use orthogonality with respect to the non-degenerate symmetric bilinear form

$$(A, B) \in M_n(\mathbb{F})^2 \mapsto \text{tr}(AB).$$

Note that if we take a linear subspace  $V$  of  $M_n(\mathbb{F})$  and an invertible matrix  $P \in \text{GL}_n(\mathbb{F})$ , the orthogonal complement of  $PVP^{-1}$  is  $PV^\perp P^{-1}$ .

Now, let  $V$  be a diagonalizable subspace of  $M_n(\mathbb{F})$  with  $\dim V = \binom{n+1}{2}$ .

The proof has four basic steps.

**Step 1.** *The subspace  $V$  contains the identity matrix  $I_n$ .*

*Proof.* We note that  $V' := V + \mathbb{F}I_n$  is a diagonalizable subspace that includes  $V$ . Statement (a) from Theorem 2 yields  $\dim V' \leq \binom{n+1}{2} = \dim V$ , and hence  $V' = V$ . It follows that  $I_n \in V$ .  $\square$

In the next two steps, we prove that  $V^\perp$  is an irreducible trivial spectrum subspace of  $M_n(\mathbb{F})$ .

**Step 2.** *The space  $V^\perp$  is irreducible.*

*Proof.* Assume that the contrary holds. Then,  $n \geq 2$  and by replacing  $V$  with a well-chosen similar subspace, we see that no generality is lost in assuming that we have found an integer  $p \in \llbracket 1, n-1 \rrbracket$  such that every matrix of  $V^\perp$  reads

$$\begin{bmatrix} [?]_{p \times p} & [?]_{p \times (n-p)} \\ [0]_{(n-p) \times p} & [?]_{(n-p) \times (n-p)} \end{bmatrix}.$$

Hence,  $V = (V^\perp)^\perp$  contains  $E_{1,n}$ , which is not diagonalizable, contradicting our assumptions.  $\square$

**Step 3.** *The space  $V^\perp$  is a trivial spectrum one.*

*Proof.* Assume on the contrary that  $V^\perp$  contains a matrix with a non-zero eigenvalue in  $\mathbb{F}$ . Replacing  $V$  with a similar subspace, we can assume that  $V^\perp$  contains a matrix of the form

$$H = \begin{bmatrix} \lambda & [?]_{1 \times (n-1)} \\ [0]_{(n-1) \times 1} & [?]_{(n-1) \times (n-1)} \end{bmatrix} \quad \text{for some } \lambda \in \mathbb{F} \setminus \{0\}.$$

We have linear mappings  $a : V \rightarrow \mathbb{F}$ ,  $C : V \rightarrow \mathbb{F}^{n-1}$ ,  $R : V \rightarrow M_{1,n-1}(\mathbb{F})$  and  $K : V \rightarrow M_{n-1}(\mathbb{F})$  such that every matrix  $M$  of  $V$  reads

$$M = \begin{bmatrix} a(M) & R(M) \\ C(M) & K(M) \end{bmatrix}.$$

Set  $W := \{M \in V : C(M) = 0\}$ . By the rank theorem,

$$\dim V = \dim C(V) + \dim W \leq (n-1) + \dim W.$$

Let  $M \in W$  be such that  $K(M) = 0$ . Since  $M$  is orthogonal to  $H$ , we get  $\lambda a(M) = 0$  and hence  $a(M) = 0$ . It follows that  $M$  is both diagonalizable and strictly upper-triangular, whence  $M = 0$ . Therefore,  $\dim K(W) = \dim W$ .

Next, let  $M \in W$ . Denoting by  $e_1$  the first vector of the standard basis of  $\mathbb{F}^n$ , we see that  $X \mapsto MX$  stabilizes  $\mathbb{F}e_1$  and the induced endomorphism on the quotient space  $\mathbb{F}^n/\mathbb{F}e_1$  is represented by  $K(M)$ . Since  $M$  is diagonalizable, so is  $K(M)$ . Hence, point (a) of Theorem 2 applies to  $K(W)$  and yields

$$\dim V \leq (n-1) + \dim K(W) \leq (n-1) + \binom{n}{2} = \binom{n+1}{2} - 1,$$

which contradicts our assumptions.  $\square$

Finally, we note that  $\dim V^\perp = n^2 - \dim V = \binom{n}{2}$ . Assume now that  $|\mathbb{F}| > 2$ . Then, Theorem 1 yields a non-isotropic matrix  $P \in \text{GL}_n(\mathbb{F})$  such that  $V^\perp = P A_n(\mathbb{F})$ , whence

$$V = (P A_n(\mathbb{F}))^\perp = S_n(\mathbb{F}) P^{-1}.$$

Since  $I_n \in V$ , the matrix  $P$  is symmetric. Note that, for all  $Q \in \text{GL}_n(\mathbb{F})$ ,

$$S_n(\mathbb{F})(QPQ^T)^{-1} = QQ^{-1}S_n(\mathbb{F})(Q^{-1})^T P^{-1}Q^{-1} = QS_n(\mathbb{F})P^{-1}Q^{-1} = QVQ^{-1}.$$

We draw two consequences from that remark:

- (1) Replacing  $V$  with a similar subspace amounts to replacing  $P$  with a congruent matrix.
- (2) If  $P$  is congruent to a scalar multiple of  $I_n$  then  $V$  is similar to  $S_n(\mathbb{F})$ .

Thus, our final step, which will complete the proof, follows:

**Step 4.** *The matrix  $P$  is congruent to a scalar multiple of  $I_n$ .*

*Proof.* The result is obvious if  $n = 1$ , so we assume that  $n \geq 2$ . Since  $P$  is symmetric and non-isotropic, it is congruent to a diagonal matrix (even if  $\mathbb{F}$  has characteristic 2). By point (1) above, no generality is lost in assuming that  $P$  is actually diagonal, with non-zero diagonal entries  $d_1, \dots, d_n$ .

Fix  $i \in \llbracket 2, n \rrbracket$ . Then,  $(E_{1,i} + E_{i,1})P^{-1} = d_i^{-1}E_{1,i} + d_1^{-1}E_{i,1}$  must be diagonalizable. Yet, its characteristic polynomial equals  $t^{n-2}(t^2 - d_i^{-1}d_1^{-1})$  and hence, for some  $\lambda \in \mathbb{F} \setminus \{0\}$ , we successively get  $d_i^{-1}d_1^{-1} = \lambda^2$  and  $d_i = d_1(d_1^{-1}\lambda^{-1})^2$  with  $d_1^{-1}\lambda^{-1} \neq 0$ . We conclude that  $P$  is congruent to  $d_1I_n$ .  $\square$

A final note on the case of the field with 2 elements, which we have left out: For subspaces of  $M_2(\mathbb{F}_2)$ , the above proof still works because the result of Theorem 1 is known to hold in that case. Yet, over  $\mathbb{F}_2$  the symmetric matrix  $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$  is non-diagonalizable because it is nilpotent and non-zero. Hence,  $\dim V < 3$  for every diagonalizable subspace of  $M_2(\mathbb{F}_2)$ . From there, an induction shows that  $\dim V < \binom{n+1}{2}$  whenever  $V$  is a diagonalizable subspace of  $M_n(\mathbb{F}_2)$  with  $n \geq 2$  (with the notation from the proof of Step 3, one applies the induction hypothesis to  $K(W)$  and one notes that  $W \cap \text{Ker } K \cap \text{Ker } a = \{0\}$ ).

## References

- [1] M. Gerstenhaber, On Nilalgebras and Linear Varieties of Nilpotent Matrices I, Amer. J. Math. **80** (1958) 614–622.
- [2] T.S. Motzkin, O. Taussky, Pairs of matrices with property L II, Trans. Amer. Math. Soc. **80** (1955) 387–401.
- [3] B. Randé, C. de Seguins Pazzis, The linear preservers of real diagonalizable matrices, Linear Algebra Appl. **435** (2011) 1257–1266.

- [4] C. de Seguins Pazzis, From primitive spaces of bounded rank matrices to a generalized Gerstenhaber theorem, *Quart. J. Math.* **65** (2014) 319–325.
- [5] C. de Seguins Pazzis, Large affine spaces of non-singular matrices, *Trans. Amer. Math. Soc.* **365** (2013) 2569–2596.
- [6] W. Waterhouse, Self-adjoint operators and formally real fields, *Duke Math. J.* **43** (1976) 237–243.